Exam Calculus 1

8 november 2010, 9.00-12.00.

This exam has 8 problems. The maximum score pre problem can be found below. Write on each page your name and student number, and on the first page your seminar group. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by arguments and/or work. Success.

1. (a) Formulate the principle of mathematical induction.

   (b) Prove that if $n \geq 1$ is a positive integer, then

$$1 \cdot 2 \cdot 3 \, + \, 2 \cdot 3 \cdot 4 + \cdots \cdot + n(n+1)(n+2) \; = \; \frac{n(n+1)(n+2)(n+3)}{4}$$

2. Find all (complex) solutions of

$$2z^4 + 2z^2 + 1 = 0$$

   and plot them in the complex plane.

3. (a) Give the following definition:
   The derivative of a function f at a point a is defined by

$$f'(a) \; \stackrel{\text{def}}{=} \; \cdots$$

   (b) We assume that $f(x)$ is differentiable on $-\infty < x < \infty$.
   Give the following definition:
   The derivative f' is continuous at a point a if ...

   (c) A function f is called continuously differentiable on $(-\infty, \infty)$, if $f(x)$ is differentiable for all $x \in (-\infty, \infty)$ and the derivative $f'(x)$ is continuous for all $x \in (-\infty, \infty)$.
   Assume that f and g are continuously differentiable on $(-\infty, \infty)$ and $g'(a) \neq 0$, $f(a) = g(a) = 0$. Show that (l'Hospitals rule is correct):

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \; = \; \frac{f'(a)}{g'(a)} \; = \; \lim_{x \to a} \frac{f(x)}{g(x)}$$

   (d) Evaluate

$$\lim_{x \to 0} \frac{e^x - 1}{x} \qquad \text{and} \qquad \lim_{x \to \infty} \left(x \, e^{1/x} - x \right)$$

4. (a) The function $f(x)$ is defined on $-\infty < x < \infty$. Give the ϵ-δ-definition of

$$\lim_{x \to a} f(x) \; = \; L.$$

(b) Prove (with the help of this definition) that

$$\lim_{x \to 2} \left(x^2 - 4x + 5\right) = 1$$

5. Newton's law of Gravitation says that the magnitude F of the force exerted by the earth on a body with unit mass at a distance r from the center of the earth is given by

$$F(r) = \begin{cases} GMr/R^3 & \text{if } r < R \\ GM/r^2 & \text{if } r \geq R \end{cases}$$

where R denotes the radius of the earth, M is the mass of the earth and G is the gravitational constant. (Note: R, M and G are constant).

(a) Is the function $F(r)$ continuous?

(b) Is F differentiable at $r = R$?

6. (a) Find the derivative of $f(x) = (1 + x^2)^x$.

(b) The function $g(x)$ is continuous on $x \geq 0$. Futhermore,

$$\int_0^{x^2} g(t)\, dt = (1 + x^2)^x$$

Evaluate $g(1)$.

7. (a) Evaluate

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx$$

(b) Evalute

$$\int \frac{\cos x}{\sqrt{1 + \sin x}}\, dx$$

(c) Evaluate

$$\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^4}\, dx$$

8. Find the solution $y(x)$ of the differential equation

$$(x^2 + 1)y' = xy$$

that satisfies $y(1) = 1$.

Maximum score:

1a	0.5	2	1.0	3a	0.3	4a	0.5	5a	0.5	6a	0.5	7a	0.5	8	1.0
b	0.5			b	0.2	b	0.5	b	0.5	b	0.5	b	0.5		
				c	0.5							c	0.5		
				d	0.5										